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Algebraic solution for the Natanzon confluent potentials*

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Abstract. An algebraic method is used to solve the bound state problem for the most general Natanzon potential for which the Schrödinger equation can be reduced to the confluent hypergeometric form (hence, *confluent potentials*) The solution is obtained straightforwardly A simple argument is given to sustain that the Natanzon potential is the most general confluent potential

1. Introduction

Last year a remarkable set of potentials was solved using an algebraic method [1]. On the other hand, some years ago Natanzon [2] found a wide family of potentials for which the Schrödinger equation can be solved. With these potentials the Schrödinger equation can be reduced either to hypergeometric form or to confluent hypergeometric form. The potentials in the second case will be called *confluent potentials.*

Cooper *et al* [3] have made a complete study of the whole family of Natanzon potentials in connection with supersymmetric quantum mechanics and they have found new solvable potentials. See also the references cited in this interesting paper. We follow much of their notation.

The present article deals with the Natanzon family of confluent potentials. It is shown that there is a simple algebraic method [4–6] to solve the bound state problem for them. It will be called *spectrum generating algebra* (or SGA) method.

The confluent Natanzon potentials

$$V(r)=\frac{g_2h^2+g_1h+\eta}{R}+\frac{\sigma_1h-\sigma_2h^2}{R^2}-\frac{5}{4}\frac{\Delta h^2}{R^3} \tag{1.1}$$

are defined in terms of six parameters g_1, g_2, σ_1, σ_2, c_0 and η and a function $h(r)$ satisfying

$$\frac{\mathrm{d}h}{\mathrm{d}r}=\frac{2h}{\sqrt{R}} \tag{1.2}$$

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where

$$R = \sigma_2 h^2 + \sigma_1 h + c_0 \tag{1.3}$$

and

$$\Delta = \sigma_1^2 - 4\sigma_2 c_0. \tag{1.4}$$

Particular cases are well known potentials. the three-dimensional harmonic oscillator ($\sigma_2 = 0$, $c_0 = 0$); the three-dimensional Coulomb potential ($\sigma_1 = 0$, $c_0 = 0$) and the Morse potential ($\sigma_2 = 0$, $\sigma_1 = 0$).

The algebraic method summarized in section 2 has in the past allowed one to solve collectively the harmonic and Coulomb potentials, the bound state problem for the Morse potential and it has been applied also to deal with the Klein–Gordon and Dirac equations [6]. In [4] a family of potentials which can now be identified with the confluent Natanzon subclass $c_0 = 0$ was solved. Details to sustain the last statement are given at the end of section 2. The results of [1] already cited can now be described as giving the solution to several potentials which are subclasses of the Natanzon confluent family of potentials.

Other developments [7] gave an ingenious and rather different approach that made it possible to deal directly with the hypergeometric potentials, a method that was successfully used in [8] to solve the bound state problem for many of the solvable *hypergeometric potentials*.

Besides [3] supersymmetric quantum mechanics (SUSYQM) has been used as an algebraic method to find new solvable potentials by others as in [9, 10]. SUSYQM has also been extended to the scattering sector [11]. The SUSYQM techniques lead to purely algebraic solutions when the potentials are shape invariant [12, 13]. Results on this are also found in [14]. The general Natanzon potentials are not directly solvable by these techniques precisely because they are not shape invariant [3].

Using the *potential group* approach the Morse and Pöschl-Teller potential problem (bound and scattering states) were shown to be connected with unitary representations of SU(2) (bound state sector) and SU(1, 1) ≈ SO(2, 1) (scattering sector) [15]. See also [16]. More recently it was shown that by making variable and operator transformations on shape invariant potentials it was possible to solve more general Hamiltonians and in this way they [17] obtained the solution for the general hypergeometric. Natanzon potential starting from the Pöschl–Teller potential and also starting from the 3D harmonic oscillator they solved confluent Natanzon potentials. On the Morse potential see also [18]. Quite recently a generalization of the potential group approach has been proposed [19] to deal both with the confluent and general hypergeometric related Schrödinger equations.

In all the above methods the Hamiltonian is written directly in terms of the operators of the algebraic structure involved. Typically quadratic expressions on the elements of the algebra are used. In our case, on the contrary we make an identification of the form $\mathcal{G}(r)(H-E)\Psi(r) = [aJ_0 + bJ_1]\Psi(r)$. The right-hand side is *linear on the generators* J_μ of SO(2, 1).

In section 2 the SGA method is summarized. In section 3 it is applied to find the bound state problem associated to the general confluent Natanzon potential and in section 4 a direct plausibility argument is given—within the context of our algebraic method—to suggest that potential (1 1) is the most general one for which the Schrödinger equation can be reduced to the confluent hypergeometric form.

2. Review of the algebraic method

In what follows a sketch of the SGA method is presented. In this approach [4, 5] the eigenvalue problem $(H-E)\Psi=0$ can be restated in terms of a particular realization of the SO(2, 1) generators J_μ as:

$$[2(1+\beta)J_0+2(1-\beta)J_1-\delta]\Psi=0 \tag{2.1}$$

where the generators satisfy the commutation relations $[J_0, J_1]=iJ_2$, $[J_1, iJ_2]=J_0$, $[J_0, iJ_2]=J_1$ and $\beta>0$ to guarantee that the operator acting on Ψ in (2.1) is compact (i.e., it has a discrete spectrum). The coefficients β and δ are determined by requiring that (2.1) reproduces the Schrödinger equation up to a common factor. In [4] the connection between different realizations of the algebra SO(2, 1) in terms of differential operators and the Schrödinger equation with several potentials was established.

It was shown that in all cases the suitable representation is $D^{(+)}$, consequently, the compact generator J_0 has the spectrum

$$j_0=\nu+\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+Q}=\nu+\frac{\gamma}{2} \tag{2.2}$$

where $Q=J_0^2-J_1^2-J_2^2$ is the Casimir operator of the algebra and $\nu=0, 1, 2, \ldots$ and in every irreducible representation γ is defined such that the value Q is

$$Q=\frac{\gamma}{2}\left(\frac{\gamma}{2}-1\right) \qquad \text{or} \qquad \gamma=1+\sqrt{1+4Q}. \tag{2.3}$$

Since γ gives the same value for Q as $\gamma'=1-\gamma$ its definition is completed requiring that $\gamma\geqslant 1$.

If the tilted operator

$$\tilde{J}_0=\exp[i\theta J_2]J_0\exp[-i\theta J_2] \tag{2.4}$$

with

$$\tanh\theta=\frac{\beta-1}{\beta+1} \tag{2.5}$$

is used in (2.1), then that equation becomes,

$$\tilde{J}_0\Psi_{\nu,\gamma}=\frac{\delta}{4\sqrt{\beta}}\Psi_{\nu,\gamma} \tag{2.6}$$

where $\Psi_{\nu,\gamma}$ is the Schrödinger wavefunction. From the above expressions it follows that

$$\gamma=-2\nu+\frac{\delta}{2\sqrt{\beta}}. \tag{2.7}$$

For the family of confluent potentials the following realization [5] for the generators J_μ is useful in terms of an arbitrary function $h(r)$,

$$\begin{aligned}
J_0&=-\frac{h}{h'^2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{h'''h}{2h'^3}+\frac{3}{4}\frac{h''^2h}{h'^4}+\frac{Q}{h}+\frac{h}{4}\\
J_1&=-\frac{h}{h'^2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{h'''h}{2h'^3}+\frac{3}{4}\frac{h''^2h}{h'^4}+\frac{Q}{h}-\frac{h}{4}\\
iJ_2&=\frac{h}{h'}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{2}\frac{h''h}{h'^2}.
\end{aligned} \tag{2.8}$$

The comparison of (2.1) with the Schrödinger equation for spherically symmetric potentials and using (2.8) for the generators, yields the following relationship between our basic function $h(r)$ and the potential $V(r)$,

$$E - V(r) = \frac{1}{2}\frac{h'''}{h'} - \frac{3}{4}\left(\frac{h''}{h'}\right)^2 - Q\left(\frac{h'}{h}\right)^2 - \frac{\beta}{4}h'^2 + \frac{\delta}{4}\frac{h'^2}{h}. \tag{2.9}$$

Equations (2.7) and (2.9) are the practical basis to apply the SGA method besides the relation (2.3) between Q and γ.

Example. Take $h(r) = 2r$ and $V(r) = -e^2/r + l(l+1)/r^2$. From (2.9) it directly follows that $\delta = 2e^2$, $\beta = -E$ and $Q = l(l+1)$. The last of these relations implies that $\gamma = 2l+2$ which we use in (2.7) to obtain that

$$E_\nu = -\frac{e^4}{4(l+\nu+1)^2}.$$

This is all the effort it takes to obtain an energy spectrum. Notice that obtaining the energy spectrum is a purely algebraic task.

Equation (2.9) makes it evident that at least one of the constants γ (hence Q), β or δ depend on the energy eigenvalue E_ν. Consequently, from now on they will be denoted γ_ν, β_ν, δ_ν and Q_ν.

The carrier space for the representation (2.8) is expanded by the functions

$$\Psi_{\nu,\gamma_\nu}(h(r)) = \frac{h^{\gamma_\nu/2}(r)}{\sqrt{h'(r)}} \exp\left[-\frac{h(r)}{2}\right] {}_1F_1(-\nu, \gamma_\nu, h(r)) \tag{2.10}$$

where ${}_1F_1(-\nu, \gamma_\nu, h(r))$ are the standard confluent hypergeometric functions. This representation space is easily derived. For example, it can be obtained directly from the self-contained section 5 of [5].

The wavefunction which appears in (2.6) is obtained by tilting the function defined in (2.10). It is amazing to observe that the only effect of the tilt is to produce again a function (2.10) having an argument $f(r)$ instead of $h(r)$ where

$$f(r) = \sqrt{\beta_\nu}\, h(r). \tag{2.11}$$

Similarly the tilted generators look exactly as in (2.8) but with $f(r)$ playing the role of $h(r)$. The tilt acts as a dilatation rescaling the basic function h by a factor $\sqrt{\beta_\nu}$.

From an algebraic point of view it is equivalent to work with the representation (2.8) over the space defined by the functions (2.10) or to work with the tilted objects replacing h by f everywhere.

From the commutation relations it follows that the tilted generators have the effect:

$$\begin{aligned} \tilde{J}_+\Psi_{\nu,\gamma_\nu}(f(r)) &= (\gamma_\nu + \nu)\Psi_{\nu+1,\gamma_\nu}(f(r)) \\ \tilde{J}_-\Psi_{\nu,\gamma_\nu}(f(r)) &= \nu\Psi_{\nu-1,\gamma_\nu}(f(r)) \end{aligned} \tag{2.12}$$

where $\tilde{J}_\pm = \tilde{J}_1 \pm i\tilde{J}_2$.

We come back to the statement that in [4] (referred to as I in this paragraph) a solution to the bound state problem of the whole family of confluent Natanzon potentials with $c_0 = 0$ was given. In I the formalism is explicitly three dimensional, but it is simpler and equivalent to deal with functions of r only. The algebra generators given in I-(3.17) are essentially the same as (2.10). Instead of (1.2) in I the differential equation for h is I-(4.6), which corresponds to (1.2) with $c_0=0$. The Hamiltonian is written explicitly in I-(4.5). Checking that the potential involved corresponds to the subclass $c_0=0$ is now straightforward.

The present formalism is invariant to the simultaneous change

$$\begin{array}{ll} h\to -h & \\ g_1\to -g_1 & \delta\to-\delta \\ \sigma_1\to-\sigma_1 & \sqrt{\beta}\to-\sqrt{\beta} \end{array} \tag{2.13}$$

while γ remains unchanged. Given this freedom there is no loss of generality choosing h positive.

In (1.2) it has been assumed that h has a positive derivative, which is not true in the case of the Morse potential. Nothing changes if a minus sign is put in front of the right-hand side in (1.2) (and (3.1)) since every term in (2.9) has an even number of derivatives of h.

3. Solution to the Natanzon confluent potential problem

The way of using the algebraic method of section 2 to solve the Schrödinger problem for the confluent Natanzon potentials is now shown step by step. First notice that R satisfies,

$$\frac{\mathrm{d}R}{\mathrm{d}r}=(2\sigma_2 h+\sigma_1)\frac{2h}{\sqrt{R}} \tag{3.1}$$

and use this relation plus (1.2) to find that (2.9) becomes,

$$E_\nu-V(r)=\frac{\delta_\nu h-\beta_\nu h^2-4Q_\nu-1}{R}-\frac{4\sigma_2h^2+\sigma_1 h}{R^2}+\frac{5(\sigma_2h^2+\frac{1}{2}\sigma_1 h)^2}{R^3}. \tag{3.2}$$

If h is eliminated from (3.2) in favour of R solving the quadratic equation (1.3) (which root we choose is immaterial) and the potential (1.1) is added to (3.2), an expression which should be E_ν itself is obtained. The expression that is actually obtained however is the sum of three types of terms: r-independent terms, terms proportional to $\sqrt{4R\sigma_2-\Delta}$ and terms proportional to R^{-1}. Hence three conditions emerge

$$E_\nu=\frac{g_2-\beta_\nu}{\sigma_2} \tag{3.3a}$$

$$0=\frac{\delta_\nu+g_1}{2\sigma_2}+\frac{\sigma_1(\beta_\nu-g_2)}{2\sigma_2^2} \tag{3.3b}$$

$$0=-\frac{2c_0(\beta_\nu-g_2)-\sigma_1(\delta_\nu+g_1)}{2\sigma_2}+\frac{(g_2-\beta_\nu)\sigma_1^2\eta}{2\sigma_2^2}-1-4Q_\nu. \tag{3.3c}$$

The last two expressions represent a linear system for β_ν and δ_ν, which can be replaced back in (3.3a). Elementary manipulations bring in

$$Q_\nu=\tfrac{1}{4}(\eta-1-c_0E_\nu) \tag{3.4a}$$

$$\delta_\nu=-g_1+\sigma_1E_\nu \tag{3.4b}$$

$$\beta_\nu=g_2-\sigma_2E_\nu. \tag{3.4c}$$

Taking (3.4*a*) with Q_ν eliminated in favour of γ_ν from (2.3) renders,

$$\gamma_\nu = 1 + \sqrt{\eta - c_0 E_\nu}. \tag{3.5}$$

Demanding that the expression $\delta_\nu = 2\sqrt{\beta_\nu}(\gamma_\nu + 2\nu)$ from (2.7) is identified with (3.4*b*) it follows that

$$\frac{g_1 - \sigma_1 E_\nu}{2\sqrt{\beta_\nu}} - \gamma_\nu = 2\nu. \tag{3.6}$$

This is the expression that determines the energy spectrum of the potential defined in section 1.

The wavefunctions are the functions $\Psi(f)$ of (2.10) with argument f given by (2.11), namely,

$$\Psi_{\nu,\gamma_\nu} \propto R^{1/4} h^{(\gamma_\nu - 1)/2} \exp[-\sqrt{\beta_\nu} h/2]\, {}_1F_1(-\nu, \gamma_\nu; \sqrt{\beta_\nu} h). \tag{3.7}$$

which has the form given in [3].

4. Why the Natanzon potentials

Finally it is argued in a quite simple way that the Natanzon potential defined in section 1 is unique. The constraint, of course, is that the Schrödinger equation can be reduced to confluent hypergeometric form or—equivalently [5]—that it can be solved with the algebraic method of section 2. In this section all subindices ν are dropped.

With no loss of generality it can be assumed that

$$\frac{\mathrm{d}h}{\mathrm{d}r} = \pm\sqrt{G(h)}. \tag{4.1}$$

to rewrite equation (2.9) in the form

$$E - V(r) = \frac{1}{4G}\frac{\mathrm{d}^2 G}{\mathrm{d}r^2} - \frac{5}{16G^2}\frac{\mathrm{d}G}{\mathrm{d}r} - Q\frac{G}{h^2} + \frac{\delta}{4}\frac{G}{h} - \frac{\beta}{4}G. \tag{4.2}$$

Since the function h gives the potential through (1.1) it must be energy independent. Making a formal derivative of (4.2) with respect to E gives an equation with four algebraically independent terms,

$$1 = \frac{G}{4h}\frac{\mathrm{d}\delta}{\mathrm{d}E} - \frac{G}{4}\frac{\mathrm{d}\beta}{\mathrm{d}E} - \frac{G}{h^2}\frac{\mathrm{d}Q}{\mathrm{d}E} \tag{4.3}$$

implying that Q, δ and β depend at most linearly on the energy. Therefore the derivatives of these three quantities should be constants. Calling them $-c_0/4$, σ_1 and $-\sigma_2$ respectively the Natanzon potential of section 1 is recovered.

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